Dynamic Screening Model of the Electric Microfield Distribution

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Starting from the linearized Klimontovich equations and the pair approximation we study the probability distribution of the electric microfield. We deduce the separation of the total microfield distribution into a high and a low frequency component. We further prove that each of these components can be calculated from a model of uncorrelated, dynamically screened particles. The low frequency component, which is of prominent interest for applications, is evaluated. The results show deviations from all earlier theories.

Introduction

Measurements at a neutral point in a plasma show an electric field varying with time in magnitude and direction. If the neutral point is at rest or moving with a velocity small compared to the velocity of light and if electromagnetic radiation processes of the plasma particles can be neglected, this microfield may be described as the superposition of the Coulomb field contributions of the plasma particles.

The calculation of the full time dependence of this electric microfield is just as impossible a task as the solution of the ergodic problem. Fortunately in applications only integral or statistical quantities of this microfield are of importance. For instance for the calculation of transport properties the autocorrelation coefficient is required; and in the case of spectral line broadening theory the statistical probability distribution of the electric field plays an important role.

It is this probability distribution of the microfield which is the object of the following investigation. In particular, the application in line broadening theory requires the subdivision of this probability distribution into a low frequency and a high frequency component, the typical time scales for these components being correlated to the ion and electron fluctuation times respectively.

The earlier theories of the statistical microfield distribution have in common that they neither could justify their basic model assumptions nor the separation of the total distribution into the so-called "high and low frequency components".

Starting from the linearized Klimontovich equations it is the *aim of this paper*

— to deduce the separation into a high and low frequency component

— to justify the model of uncorrelated, dynamically screened particles

— and to present the evaluation of this *dynamic screening model*.

Formulation and Historical Review of the Problem

We define the probability density function for the electric microfield $W(\boldsymbol{E})$ by the relation

$$\mathrm{d}P_E = W(\boldsymbol{E}) \, \mathrm{d}\boldsymbol{E} \tag{1}$$

where $\mathrm{d}P_E$ is the probability that we observe the electric field at our neutral point in the range $\boldsymbol{E}$ and $\boldsymbol{E} + \mathrm{d}\boldsymbol{E}$.

For a system of N electrons and N ions this probability density can be formulated through

$$W_{2N}(\boldsymbol{E}) = \int \ldots \int \delta \left[\boldsymbol{E} - \sum_{\substack{j=1 \\ \alpha = \mathrm{i,e}}}^{N} \boldsymbol{E}^{\alpha}(\boldsymbol{r}_j{}^{\alpha}) \right] P_{2N}(\boldsymbol{r}_1{}^{\mathrm{e}}, \ldots, \boldsymbol{r}_N{}^{\mathrm{i}}) \, \mathrm{d}\boldsymbol{r}_1{}^{\mathrm{e}} \ldots \mathrm{d}\boldsymbol{r}_N{}^{\mathrm{i}}. \tag{2}$$

Here $\boldsymbol{E}^{\alpha}(\boldsymbol{r}_j{}^{\alpha})$ denotes the Coulomb field of the j-th particle of the kind α ($\alpha = \mathrm{e, i}$) at the neutral point, P_{2N} is the distribution function in the configuration Γ-space, and the Dirac function cuts out that part of the Γ-space satisfying the subsidiary condition $\boldsymbol{E} = \sum \boldsymbol{E}^{\alpha}(\boldsymbol{r}_j{}^{\alpha})$. The integration covers the whole configuration Γ-space.

All earlier studies of the problem consider the case of the equilibrium plasma which specifies the distribution function P_{2N} to the Gibbs-factor

$$P_{2N}(\boldsymbol{r}_1{}^e, \ldots, \boldsymbol{r}_N{}^i) = \frac{\exp\{-\Phi(\boldsymbol{r}_1{}^e, \ldots, \boldsymbol{r}_N{}^i)/kT\}}{\int \ldots \int \exp\{-\Phi(\boldsymbol{r}_1{}^e, \ldots, \boldsymbol{r}_N{}^i)/kT\}\, d\boldsymbol{r}_1{}^e \ldots d\boldsymbol{r}_N{}^i} \tag{3}$$

where Φ is the potential energy and T the absolute temperature of the system.

Under the aspect of applicability an evaluation of Eq. (2) is of value only if at the same time the separation into a high and low frequency component is achieved. Therefore most theories * used for the calculation of these components various "dressed particle models" which allowed them to apply the formula (2) for a one component system of N particles (electrons or ions), the interaction being taken into account by introducing into Eq. (2) effective fields and an effective Gibbs-factor.

The pioneering work in this field is due to Holtsmark [2], who considered the ion microfield only and neglected correlations altogether. This allowed him an analytical solution in a simple form.

The first approach to include correlations and interactions of the particles in the calculation of the low frequency component of the electric microfield is due to Ecker and Müller [3]. They based their approach on a "dressed particle model" implying that one can omit the correlation in the Gibbs-factor, if one takes into account the screening effect of the electrons and ions on the ion Coulomb fields.

Baranger and Mozer [4] discussed in detail the necessity of a subdivision of the electric microfield into a high and low frequency component and introduced a cluster expansion method for the evaluation. They attribute the high frequency component to the electrons and the low frequency component to the ions which they consider statically screened by the electrons. The electron-electron interaction and the ion-ion interaction are taken into account through an effective pair correlation function. As pointed out by Pfennig and Trefftz [5] a numerical error in Baranger and Mozer's work requires a correction of their results.

Rand [6] considered the low frequency component using the same basic model assumption as Baranger

and Mozer. In addition to the static electron screening however he treated the ions as screened dynamically by their own kind. No justification for this "double counting" of the ion-ion interaction is given. There are no numerical results available from his work since Rand did not succeed to evaluate his model.

All theories referred to so far neglect long range collective contributions. In a paper by Hooper [7] however the model of Baranger and Mozer was evaluated using the collective variables introduced by Broyles [8]. Supposedly the long range collective effects were included in this calculation and Hooper himself interpreted the differences he found in comparison to Baranger and Mozer's data as due to the long range collective effects. However, the numerical error, detected by Pfennig and Trefftz, proved that the results of Hooper are actually identical
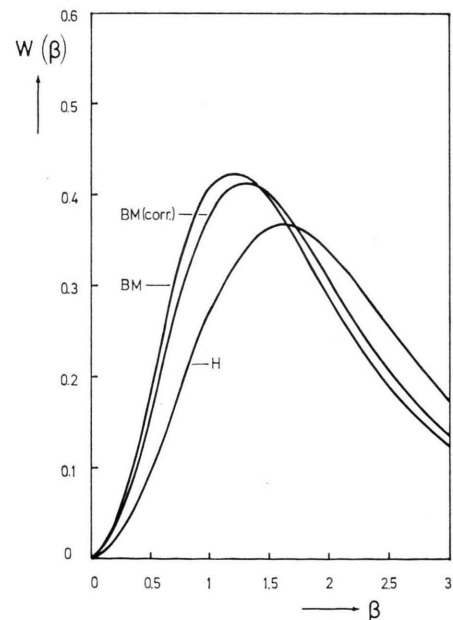
Fig. 1a. High frequency microfield distribution $W(\beta)$ plotted versus the reduced field strength $\beta = E/E_0$, where E_0 designates the field produced in the mean particle distance R_0. The curve H shows Holtsmark's data. The curves BM indicate the results of Baranger and Mozer for a typical parameter $R_0/\lambda_D = .6$, where λ_D is the Debye length. Hooper's results practically coincide with the corrected version of Baranger and Mozer's data.

* An exception is the paper by Weise [1] who calculated the total distribution function without being able to separate into a high and low frequency component.

with the corrected data of Baranger and Mozer in the high as well as in the low frequency component.

In contrast, papers by Ecker and Fischer [9] and Klein and Krall [10] show that at least for the high frequency component the effect of the long range collective contributions causes significant deviations from the Baranger and Mozer values. Thus one is forced to conclude that Hooper's method does not properly account for the long range collective contributions.

The microfield distributions of the earlier theories — quoted above — are demonstrated in Figs. 1 a and 1 b.
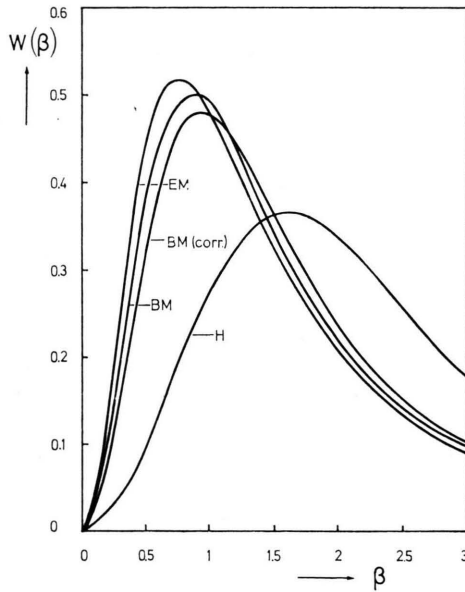

Fig. 1 b. Low frequency microfield distribution $W(\beta)$. All symbols and the typical parameter are chosen as in Figure 1 a. In addition EM indicates the results derived by Ecker and Müller. Here too the data provided by Hooper are identical with those of Baranger and Mozer.

Derivation of the Model

As we have discussed, all preceding theories of the probability distribution of the electric microfield use "ad hoc"-assumptions to separate the total distribution into a high and low frequency component and to take into account the internal interaction of the electron-ion system. We will prove here that the microfield distribution can be evaluated from a model of uncorrelated, dynamically screened "quasi-particles". We base our derivation

on two approximations frequently employed in modern plasma theory: the linearization of the Klimontovich equations and the pair-approximation **.

In the "Klimontovich-formalism" the exact distribution functions for an electron-ion system are given by

$$f_a^{Kl}(\boldsymbol{r}, \boldsymbol{v}, t) = \sum_{j=1}^{N} \delta[\boldsymbol{r} - \boldsymbol{r}_j^a(t)] \, \delta[\boldsymbol{v} - \boldsymbol{v}_j^a(t)] \,. \quad (4)$$

These functions satisfy the Klimontovich equations

$$\frac{\partial f_a^{Kl}}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_a^{Kl}}{\partial \boldsymbol{r}} + \frac{e_a}{m_a} \boldsymbol{E} \cdot \frac{\partial f_a^{Kl}}{\partial \boldsymbol{v}} = 0 \quad (5)$$

where the electric field is given by Poisson's equation

$$\nabla \cdot \boldsymbol{E} = 4\pi e \int (f_i^{Kl} - f_e^{Kl}) \, \mathrm{d}\boldsymbol{v} \,. \quad (6)$$

$[\boldsymbol{r}_j^a(t), \boldsymbol{v}_j^a(t)]$ are the coordinates, m_a the masses and e_a the charges of the electron and ions respectively.

Up to here no approximation has been involved. The Eqs. (5, 6) are exact, but their exact solution is intractable.

The approach to solve these exact microscopic equations, which has been successfully employed in modern kinetic theory [11, 12], is the linearization of (5) by substituting the Klimontovich distribution function f_a^{Kl} in the third term on the left hand side by the ensemble-averaged distribution function $f_a = \langle f_a^{Kl} \rangle$. This yields

$$\frac{\partial f_a^{Kl}}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_a^{Kl}}{\partial \boldsymbol{r}} + \frac{e_a}{m_a} \boldsymbol{E} \cdot \frac{\partial f_a}{\partial \boldsymbol{v}} = 0 \,. \quad (7)$$

Employing Fourier and Laplace transformation in the solution of Eqs. (6, 7) one finds for the time dependent microscopic electric field at the neutral point $\boldsymbol{r}_0$

$$\boldsymbol{E}(\boldsymbol{r}_0, t) = \sum_{\substack{j=1 \\ a=i,e}}^{N} \int \mathrm{d}\boldsymbol{k} \, \widetilde{\boldsymbol{E}}_j^a(\boldsymbol{k}, t) \exp\{i\boldsymbol{k} \cdot \boldsymbol{r}_0\} \quad (8)$$

with the field-Fourier-transform

$$\widetilde{\boldsymbol{E}}_j^a(\boldsymbol{k}, t) = -\frac{i\boldsymbol{k} e_a}{2\pi^2 k^2} \frac{\exp\{-i\boldsymbol{k} \cdot (\boldsymbol{r}_{j0}^a + \boldsymbol{v}_{j0}^a t)\}}{\varepsilon^+(\boldsymbol{k}, -i\boldsymbol{k} \cdot \boldsymbol{v}_{j0}^a)} + \sum_{\nu} R_{\nu}^{ja} \exp\{p_{\nu}(k) t\} \,. \quad (9)$$

Here $\varepsilon^+(\boldsymbol{k}, p)$ denotes the plasma dielectric function

$$\varepsilon^+(\boldsymbol{k}, p) = 1 - \frac{4\pi i e^2}{k^2} \sum_{a=i,e} \frac{1}{m_a} \int \frac{\boldsymbol{k} \cdot (\partial f_a/\partial \boldsymbol{v}) \, \mathrm{d}\boldsymbol{v}}{p + i\boldsymbol{k} \cdot \boldsymbol{v}} \quad (10)$$

** The use of the pair approximation, of course, means that we do not consider here the long range collective contributions.

$p_\nu(k)$ are the zeros of the dielectric function and $R_\nu^{j\alpha}$ the residues at the poles $p_\nu(k)$.

For a system in thermodynamic equilibrium f_α is the Maxwell-Boltzmann distribution. It then turns out [13] that all zeros of the dielectric function have a negative real part, so that the contribution of the second term in Eq. (9) is negligible for times $t > t_{\min}$, where $t_{\min}$ is a characteristic time constant discussed in Appendix A.

For times $t > t_{\min}$ we have then

$$\boldsymbol{E}(t) = \sum_{\substack{j=1 \\ \alpha = i, e}}^{N} \boldsymbol{E}_{\text{eff}}^\alpha (\boldsymbol{r}_{j0}^\alpha + \boldsymbol{v}_{j0}^\alpha t, \boldsymbol{v}_{j0}^\alpha) \qquad (11)$$

where

$$\boldsymbol{E}_{\text{eff}}^\alpha (\boldsymbol{r}_{j0}^\alpha + \boldsymbol{v}_{j0}^\alpha t, \boldsymbol{v}_{j0}^\alpha) \qquad (12)$$
$$= -\frac{i e_\alpha}{2 \pi^2} \int \frac{\boldsymbol{k} \exp\left\{ -i \boldsymbol{k} \cdot (\boldsymbol{r}_{j0}^\alpha + \boldsymbol{v}_{j0}^\alpha t) \right\} \mathrm{d}\boldsymbol{k}}{k^2 \, \varepsilon^+ (\boldsymbol{k}, -i \boldsymbol{k} \cdot \boldsymbol{v}_{j0}^\alpha)}$$

holds. In this formulation we have now chosen the neutral point as the origin of our coordinate system.

A formulation of $W_{2N}(\boldsymbol{E})$ more appropriate for our purposes here follows from the extension of Eq. (2) to the full phase space through a straightforward canonical transformation in the form

$$W_{2N}(\boldsymbol{E}) = \int \ldots \int \delta\left[\boldsymbol{E} - \boldsymbol{E}(t) \right] f_{2N}(\boldsymbol{r}_{10}^e, \ldots, \boldsymbol{v}_{N0}^i)$$
$$\cdot \, \mathrm{d}\boldsymbol{r}_{10}^e \ldots \mathrm{d}\boldsymbol{v}_{N0}^i \qquad (13)$$

where f_{2N} designates the distribution of the initial coordinates at $t = 0$.

We develop the distribution f_{2N} in the pair approximation

$$f_{2N} = f_e^M(\boldsymbol{v}_{10}^e) \ldots f_i^M(\boldsymbol{v}_{N0}^i)$$
$$\cdot \left\{ 1 + \sum_{\substack{j<k \\ \alpha, \alpha' = i, e}} \frac{1}{V^{2N}} \, g_2^{\alpha\alpha'}(\boldsymbol{r}_{j0}^\alpha, \boldsymbol{r}_{k0}^{\alpha'}) \right\}, \qquad (14)$$

f_e^M, f_i^M designate Maxwell distributions, V is the plasma volume.

Introducing the results (11, 14) into Eq. (13) and using Fourier transformation yields

$$\widetilde{W}_{2N}(\boldsymbol{q}) = \int \ldots \int \exp\left\{ -i \boldsymbol{q} \cdot \sum \boldsymbol{E}_{\text{eff}}^\alpha \right\} f_e^M(\boldsymbol{v}_{10}^e) \ldots f_i^M(\boldsymbol{v}_{N0}^i) \left\{ 1 + \sum \frac{1}{V^{2N}} \, g_2^{\alpha\alpha'} \right\} \mathrm{d}\boldsymbol{r}_{10}^e \ldots \mathrm{d}\boldsymbol{v}_{N0}^i. \qquad (15)$$

A closer inspection of Eq. (15) (for details see Appendix B) shows that all those terms connected with the pair correlations are damped and may again be neglected for times $t > t_{\min}$. Consequently we have

$$\widetilde{W}_{2N}(\boldsymbol{q}) = \left[\frac{1}{V} \int \int \exp\left\{ -i \boldsymbol{q} \cdot \boldsymbol{E}_{\text{eff}}^e (\boldsymbol{r}_0^e + \boldsymbol{v}_0^e t, \boldsymbol{v}_0^e) \right\} f_e^M(\boldsymbol{v}_0^e) \, \mathrm{d}\boldsymbol{r}_0^e \, \mathrm{d}\boldsymbol{v}_0^e \right]^N$$
$$\times \left[\frac{1}{V} \int \int \exp\left\{ -i \boldsymbol{q} \cdot \boldsymbol{E}_{\text{eff}}^i (\boldsymbol{r}_0^i + \boldsymbol{v}_0^i t, \boldsymbol{v}_0^i) \right\} f_i^M(\boldsymbol{v}_0^i) \, \mathrm{d}\boldsymbol{r}_0^i \, \mathrm{d}\boldsymbol{v}_0^i \right]^N. \qquad (16)$$

The result allows two conclusions:

1. We see that the Fourier transform of the microfield distribution presents itself as the product of two separate contributions, one of them depending only on the electron, the other one only on the ion coordinates and velocities. This means that the total microfield distribution is given by the superposition of two components, one of them varying on the timescale of the electron fluctuations $\tau_e = R_0/V_{\text{th}}^e$, the other one on the timescale of the ion fluctuations $\tau_i = R_0/V_{\text{th}}^i$ (R_0 being the mean particle distance and V_{th} the thermal velocity). These two components are of course the high and the low frequency component. Note that this result does not mean that electrons or ions contribute only to the high or low frequency component respectively. In fact, via the screening effects both particle kinds in general affect both components.

2. We further see that each of these components can be described by a "dressed particle model" of uncorrelated, dynamically screened electrons or ions respectively. Of course, the screening contribution of the ions in the high frequency component is negligible in comparison to that of the electrons.

We emphasize that these two basic conclusions are the results of our derivation and not an assumption postulated to formulate our model.

Evaluation of the Model

We consider the numerical evaluation of our model for the low frequency component which is of prominent interest for applications.

Applying consequently the variable transformation $\boldsymbol{r} = \boldsymbol{r}_0^i + \boldsymbol{v}_0^i t$ to the second factor of Eq. (16) we have in the limit $N \to \infty$, $V \to \infty$, $n = N/V =$

const for the low frequency component

$$\widetilde{W}_{\text{lf}}(\boldsymbol{q}) = \exp\{n\, h_1(\boldsymbol{q})\} \tag{17}$$

with

$$h_1(\boldsymbol{q}) = \iint d\boldsymbol{r}\, d\boldsymbol{v}\, f_{\text{i}}^{\text{M}}(\boldsymbol{v})\, [\exp\{-i\,\boldsymbol{q}\cdot\boldsymbol{E}_{\text{eff}}^{\text{i}}(\boldsymbol{r},\boldsymbol{v})\} - 1]\,. \tag{18}$$

An exact analytical solution of this integral presents formidable difficulties. We therefore expand the integrand about a velocity independent Debye field

$$\boldsymbol{E}^{(0)}(\boldsymbol{r},\xi) = \frac{\text{e}}{r^3}\,\boldsymbol{r}\left(1 + \frac{\xi}{\lambda_{\text{D}}}\,r\right)\exp\left\{-\frac{\xi}{\lambda_{\text{D}}}\,r\right\} \tag{19}$$

where ξ is a parameter available to be adjusted such that the following expansion

$$\begin{aligned}
1 - \exp\{-i\,\boldsymbol{q}\cdot\boldsymbol{E}_{\text{eff}}^{\text{i}}(\boldsymbol{r},\boldsymbol{v})\} &\approx 1 - \exp\{-i\,\boldsymbol{q}\cdot\boldsymbol{E}^{(0)}(\boldsymbol{r},\xi)\}\\
&+ i\,\boldsymbol{q}\cdot[\boldsymbol{E}_{\text{eff}}^{\text{i}}(\boldsymbol{r},\boldsymbol{v}) - \boldsymbol{E}^{(0)}(\boldsymbol{r},\xi)]\,\exp\{-i\,\boldsymbol{q}\cdot\boldsymbol{E}^{(0)}(\boldsymbol{r},\xi)\}
\end{aligned} \tag{20}$$

shows optimal convergence, which is the case if

$$\left|\,\boldsymbol{q}\cdot[\boldsymbol{E}_{\text{eff}}^{\text{i}}(\boldsymbol{r},\boldsymbol{v}) - \boldsymbol{E}^{(0)}(\boldsymbol{r},\xi)]\,\right| \ll 1 \tag{21}$$

is secured in the whole range of interest. This condition can only be met if the number of particles in the Debye region $\delta = (R_0/\lambda_{\text{D}})^{-3}$ is large.

However this means no additional restriction of our work, since this requirement is already a necessary condition for the pair approximation to hold. The optimal value for the parameter ξ turns out to be $\xi = (1.5)^{1/2}$.

Within these limits we get for the spectral function

$$\begin{aligned}
\widetilde{W}_{\text{lf}}(\boldsymbol{q}) &= \exp\left\{-n\int d\boldsymbol{r}\,[1 - \exp\{-i\,\boldsymbol{q}\cdot\boldsymbol{E}^{(0)}(\boldsymbol{r},\xi)\}]\right\}\\
&\times \exp\left\{-n\iint d\boldsymbol{r}\,d\boldsymbol{v}\,i\,\boldsymbol{q}\cdot[\boldsymbol{E}_{\text{eff}}^{\text{i}} - \boldsymbol{E}^{(0)}]\exp\{-i\,\boldsymbol{q}\cdot\boldsymbol{E}^{(0)}\}\,f_{\text{i}}^{\text{M}}(\boldsymbol{v})\right\} =: \exp\left\{-x^{3/2}\,[\psi_0(y,\xi) - \psi_{\text{corr}}(y,\xi)]\right\}
\end{aligned} \tag{22}$$

where we have introduced the dimensionless variables

$$x = |\,\boldsymbol{q}\,|\,E_0\,; \quad y = \sqrt{x}\,R_0/\lambda_{\text{D}} \tag{23}$$

with

$$\begin{aligned}
&\frac{4}{15}\,(2\,\pi)^{3/2}\,n\,R_0^3 = 1\,; \quad E_0 = e/R_0^2\,;\\
&\quad\quad \lambda_{\text{D}}^2 = (4\,\pi\,n\,e^2/k\,T)^{-1}\,.
\end{aligned} \tag{24}$$

The first factor in (22) may be reduced to functions already calculated, since we have

$$\psi_0(y,\xi) = \psi_{\text{E}}(\xi y/\sqrt{2}) = \psi_{\text{BM1}}(\xi y) \tag{25}$$

where ψ_{E}, ψ_{BM1} are the corresponding results of Ecker and Müller and Baranger and Mozer (without ion-ion correlations) respectively. After some transformation we get for the correction term ψ_{corr} an expression which is integrated numerically.

A graphical representation of our results for the distribution function of the reduced field $\beta = E/E_0$ is given in Fig. 2 for characteristic values of R_0/λ_{D}.
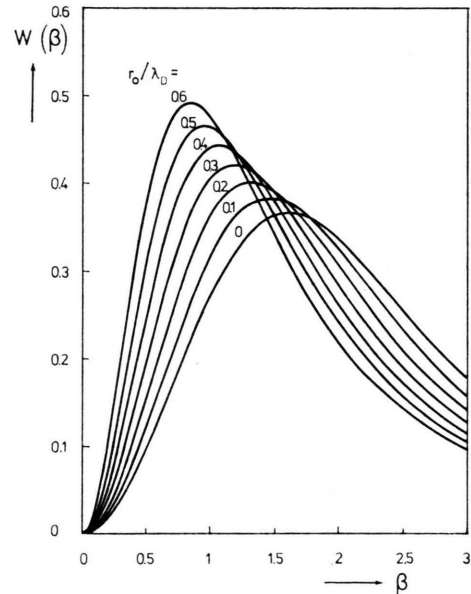

Fig. 2. Our results for the low frequency component of the microfield distribution for various values of the typical parameter R_0/λ_{D}.

Summary and Discussion

Starting from the linearized Klimontovich equations we showed that the total microfield distribution separates quite naturally into a high and low frequency component, a fact, which had been postulated before.

We further proved that each of these components can be calculated from a *model of uncorrelated, dynamically screened particles.*

"Dressed particle models" have been used in the literature before on the basis of "ad hoc"-assumption. All these models differ from the one deduced here.

In the high frequency component we have in contrast to Baranger and Mozer instead of pair-correlated Coulomb-particles uncorrelated, dynamically screened electrons. Moreover we find in principle screening contributions of the ions, which however affect the results only little due to the large inertia of the ions.

In the low frequency component all previous models account for the electron-ion correlation by static electron screening; the calculation described above requires dynamic screening which however in practice yields the same effect. The ion-ion correlations in our model are accounted for by a further shielding of the effective field. This is similar to the work of Ecker and Müller, however with the difference – which is essential here – that our shielding is dynamic and not static. It is different from the approach of Baranger and Mozer who accounted for the ion-ion interaction by an effective pair correlation. It is also different from the model of Rand who double counts the ion-ion correlations through the effective fields and the pair correlation functions.

The quantitative comparison of our results for the low frequency component with those of Ecker and Müller and Baranger and Mozer respectively is shown for the typical parameter $R_0/\lambda_D = 0.6$ in Figure 3.

Compared to the data of Ecker and Müller our distribution is shifted to larger field values, compared to the data of Baranger and Mozer, it is shifted to lower values. The first statement appears natural since static shielding represents a stronger ion-ion interaction than dynamic screening, which approaches zero for large ion velocities. The difference to the distribution of Baranger and Mozer in-
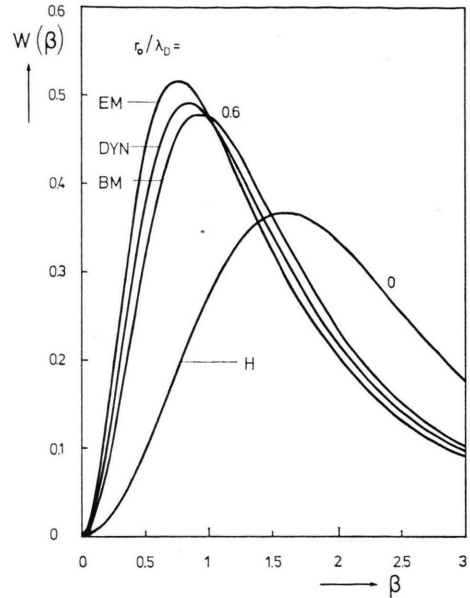

Fig. 3. Comparison of the low frequency component data as calculated by Ecker and Müller (EM) and Baranger and Mozer in corrected form (BM) with the results of the *dynamic screening model* derived in this paper (DYN). All other symbols and the typical parameter are chosen as in Figure 1.

dicates that their cluster expansion underestimates the ion-ion correlations.

One more comment: the correction term ψ_{corr} in Eq. (22) is practically zero for $\xi = (1.5)^{1/2}$. This means that we get numerically the same results for the microfield distribution if we take instead of our *dynamic screening model* the model of Ecker and Müller (uncorrelated, statically screened ions) and apply the characteristic shielding length

$$(\lambda_D')^{-2} = 4\,\pi\,e^2\,(n_e + \tfrac{1}{2}\,n_i)/k\,T = (\sqrt{1.5}\,\lambda_D)^{-2}\,.$$

(26)

We emphasize however that the factor $1/2$ in the ion density arises from the dynamic screening effects and is not due to a double counting of the ion-ion interaction in the work quoted.

Appendix A

Here we will estimate the order of magnitude of the characteristic time constant $t_{\min}$, defined on page 416, and discuss its significance for a system in thermodynamic equilibrium.

The additional terms in Eq. (9) and the terms connected with the pair correlation function in Eq. (15) have the form

$$S_\nu = \bar{S}_\nu \exp\{p_\nu(k)\,t\}$$

(A 1)

where $p_\nu(k)$ are the zeros of $\varepsilon^+(\boldsymbol{k}, p)$. For an electron-ion system in thermodynamic equilibrium Fried and Gould [13] showed that with the exception of the Landau branch the contributions of all these terms may be neglected for times $t \gg 1/\omega_{pi}$.

The Landau branch has the smallest damping decrement which even goes to zero for $k \to 0$. The over-all contribution of the small k-value region however was considered by Guernsey [14] who showed that it decreases like $1/(\omega_{pe} t)^{1/2}$. Due to $\omega_{pe} \gg \omega_{pi}$ the neglection of these "Landau contributions" for $t > t_{\min}$ is therefore also justified if we chose $t_{\min} \gg 1/\omega_{pi}$.

A comment is in place with respect to the fact that we are considering time scales in an equilibrium system:

The occurrence of the time in the microscopic description of an equilibrium system is no surprise. On the other hand macroscopic or time averaged quantities of such a system should not exhibit time dependence. Therefore we should not expect time aspects for our microfield distribution.

The condition $t > t_{\min}$ is no contradiction to this expectation. The probability distribution of the electric microfield follows from our calculation time independent for all times. The only difference is that for $t > t_{\min}$ it can be calculated from a system of uncorrelated, dynamically screened particles, in the range $t < t_{\min}$ the same result presents itself in a different mathematical form.

Appendix B

We study here the damping of all those terms in Eq. (15) connected with the pair correlations. With the power expansion of the exponential function

$$\exp\left\{-i\,\boldsymbol{q} \cdot \sum_{\substack{j=1 \\ a=i,e}}^{N} \boldsymbol{E}_{\text{eff}}^\alpha(j)\right\} = 1 - i\,\boldsymbol{q} \cdot \sum_{\substack{j=1 \\ a=i,e}}^{N} \boldsymbol{E}_{\text{eff}}^\alpha(j) + \frac{1}{2} \sum_{\substack{j=1 \\ a=i,e}}^{N} \sum_{\substack{l=1 \\ a'=i,e}}^{N} i\,\boldsymbol{q} \cdot \boldsymbol{E}_{\text{eff}}^\alpha(j)\, i\,\boldsymbol{q} \cdot \boldsymbol{E}_{\text{eff}}^{\alpha'}(l) - \ldots + \ldots \quad \text{(B 1)}$$

these terms can be cast into the form

$$I_{rs} = \int d\boldsymbol{r}_{j0}^\alpha \int d\boldsymbol{v}_{j0}^\alpha \int d\boldsymbol{r}_{k0}^{\alpha'} \int d\boldsymbol{v}_{k0}^{\alpha'} \, f_\alpha^M(\boldsymbol{v}_{j0}^\alpha)\, f_{\alpha'}^M(\boldsymbol{v}_{k0}^{\alpha'}) \, g_2^{\alpha\alpha'}(\boldsymbol{r}_{j0}^\alpha, \boldsymbol{r}_{k0}^{\alpha'})\, [i\,\boldsymbol{q} \cdot \boldsymbol{E}_{\text{eff}}^\alpha(j)]^r [i\,\boldsymbol{q} \cdot \boldsymbol{E}_{\text{eff}}^{\alpha'}(k)]^s \quad \text{(B 2)}$$

with $r \geqq 1$, $s \geqq 1$. For the case $r = 1$, $s = 1$ Montgomery [12] showed the damping of this expression.

For a homogeneous system, the pair correlation function $g_2^{\alpha\alpha'}$ depends only on the relative distance $\boldsymbol{r}_{j0}^\alpha - \boldsymbol{r}_{k0}^{\alpha'}$ so that we may write

$$g_2^{\alpha\alpha'}(\boldsymbol{r}_{j0}^\alpha, \boldsymbol{r}_{k0}^{\alpha'}) = \int \tilde{g}_2^{\alpha\alpha'}(\boldsymbol{k}_g) \exp\left\{i\,\boldsymbol{k}_g \cdot (\boldsymbol{r}_{j0}^\alpha - \boldsymbol{r}_{k0}^{\alpha'})\right\} d\boldsymbol{k}_g. \quad \text{(B 3)}$$

Inserting (B 3) and (12) into (B 2) yields after some transformation

$$I_{rs} = \int \left[\prod_{\varrho \geqq 2} d\boldsymbol{k}_\sigma \prod_{\sigma \geqq 2} d\boldsymbol{k}_\sigma' \right] \int d\boldsymbol{k}^* \prod_{\varrho \geqq 2}^{r} \frac{\boldsymbol{q} \cdot \boldsymbol{k}_\varrho}{k_\varrho^2} \prod_{\sigma \geqq 2}^{s} \frac{\boldsymbol{q} \cdot \boldsymbol{k}_\sigma'}{k_\sigma'^2} \frac{\boldsymbol{q} \cdot (\boldsymbol{k}^* - \sum_{\varrho \geqq 2} \boldsymbol{k}_\varrho)}{|\boldsymbol{k}^* - \sum_{\varrho \geqq 2} \boldsymbol{k}_\varrho|^2} \frac{\boldsymbol{q} \cdot (-\boldsymbol{k}^* - \sum_{\sigma \geqq 2} \boldsymbol{k}_\sigma')}{|-\boldsymbol{k}^* - \sum_{\sigma \geqq 2} \boldsymbol{k}_\sigma'|^2} \, \tilde{g}_2(\boldsymbol{k}^*)\, I_{v_1} I_{v_2} \quad \text{(B 4)}$$

with the abbreviation

$$I_{v_1} = \int d\boldsymbol{v}_1 \frac{\exp\left\{-i\,\boldsymbol{k}^* \cdot \boldsymbol{v}_1 t\right\}}{\varepsilon^+[\boldsymbol{k}^* - \sum_{\varrho \geqq 2} \boldsymbol{k}_\varrho, -i(\boldsymbol{k}^* - \sum_{\varrho \geqq 2} \boldsymbol{k}_\varrho) \cdot \boldsymbol{v}_1]} f^M(\boldsymbol{v}_1) \prod_{\varrho \geqq 2} \frac{1}{\varepsilon^+(\boldsymbol{k}_\varrho, -i\,\boldsymbol{k}_\varrho \cdot \boldsymbol{v}_1)} \quad \text{(B 5)}$$

and a corresponding expression for I_{v_2}. In the case $\boldsymbol{k}^* = 0$ the whole integral (B 4) disappears identically.

For the $\boldsymbol{v}_1$-integration we chose the v_x-axis in the direction of the $\boldsymbol{k}^*$-vector and transform to the new variable $p = -i|\boldsymbol{k}^*| v_x$. Applying the calculus of residues to this integral one may show that we get only terms like

$$S_\nu{}^\varrho = S_\nu{}^\varrho(\bar{p}_{0\nu}^\varrho, \boldsymbol{k}^*, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_\varrho) \exp\left\{\bar{p}_{0\nu}^\varrho t\right\} \qquad \text{with } \text{Re}(\bar{p}_{0\nu}^\varrho) < 0. \quad \text{(B 6)}$$

For the damping decrement $\bar{\gamma}^0_{0\nu}$ one finds

$$\bar{\gamma}^0_{0\nu} := -\operatorname{Re}(\bar{p}^0_{0\nu}) = \frac{k^*}{\bar{k}_x} \gamma_\nu (|\boldsymbol{k}|) \qquad (\text{B 7})$$

where $\boldsymbol{k}$ is either $\boldsymbol{k}^* - \Sigma\,\boldsymbol{k}_\varrho$ or $\boldsymbol{k}_\varrho$, and γ_ν denotes the well known Landau damping decrement.

The case $\boldsymbol{k}^* = 0$ has been already excluded. For $\boldsymbol{k}^* \neq 0$, knowing that γ_ν is a monotonous function of k, it suffices to consider the smallest value for $k \to 0$. In this case we refer to Appendix A where we have shown that sufficient damping occurs for $t > t_{\min}$.

Similar arguments hold for the v_y and v_z-integration and for the $\boldsymbol{v}_2$-integral. For $t > t_{\min}$ we have therefore no contributions to Eq. (15) from the pair correlation function.

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